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Geometry of the Submanifolds of SEX_n. II. The Generalized Fundamental Equations for the Hypersubmanifold of SEX_n

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A connection which is both Einstein and semisymmetric is called an SE connection, and a generalized n-dimensional Riemannian manifold on which the differential geometric structure is imposed by $g_{\lambda\mu}$ through an SE connection is called an n-dimensional SE manifold and denoted by SEX_n. This paper is a direct continuation of earlier work. In this paper, we derive the generalized fundamental equations for the hypersubmanifold of SEX_n, including generalized Gauss formulas, generalized Weingarten equations, and generalized Gauss-Codazzi equations.

1. PRELIMINARIES

This paper is a direct continuation of Chung *et al.* (1989), which will be denoted by I in further considerations in the present paper. It is based on the results and symbolism of I. Whenever necessary, these results will be quoted in the text.

Let SEX_n be an n-dimensional SE manifold connected by an SE connection $\Gamma^{\nu}_{\lambda\mu}$. Let X_{n-1} be the hypersubmanifold of SEX_n connected by the induced connection Γ^{k}_{ij} of $\Gamma^{\nu}_{\lambda\mu}$ on SEX_n. In virtue of I, Remark 4.3, X_{n-1} *is also an SE manifold.*

Since $m = n-1$ in our case, there exists only one unit normal N^{α} to X_{n-1} satisfying [(I.3.5)]

$$h_{\alpha\beta}B_i^{\alpha}N^{\beta} = N_{\alpha}B_i^{\alpha} = 0, \qquad h_{\alpha\beta}N^{\alpha}N^{\beta} = 1 \tag{1.1}$$

Therefore, some results obtained in I should be revised. In the following we list those revised results which are necessary in the present paper.

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The tensor B_λ^ν satisfies the following identities [(I.3.19)]:

$$B_\lambda^\nu = \delta_\lambda^\nu - N_\lambda N^\nu \tag{1.2a}$$

$$B_\lambda^\alpha N_\alpha = B_\alpha^\nu N^\alpha = 0 \tag{1.2b}$$

The symmetric and skew-symmetric parts of the induced metric tensor g_{ij} on X_{n-1} of $g_{\lambda\mu}$ in SEX$_n$ are given by [(I.3.26)]

$$h_{ij} = h_{\alpha\beta} B_i^\alpha B_j^\beta, \qquad k_{ij} = k_{\alpha\beta} B_i^\alpha B_j^\beta \tag{1.3}$$

In virtue of the condition (I.3.28), there exists a unique tensor h^{ik} defined by $h_{ij}h^{ik} = \delta_j^k$, and the tensors h_{ij} and h^{ij} may be used for raising and/or lowering indices of the induced tensors on X_{n-1} in the usual manner (I, Theorem 3.11b). However, the reverse relations of (1.3) may be given by [(I.3.30)]

$$h_{\lambda\mu} = h_{ij} B_\lambda^i B_\mu^j + N_\lambda N_\mu \tag{1.4a}$$

$$h^{\lambda\nu} = h^{ij} B_i^\lambda B_j^\nu + N^\lambda N^\nu \tag{1.4b}$$

Let Ω_{ij} be the generalized coefficients of the second fundamental form of X_{n-1} and $\overset{0}{D}_j$ be the symbolic vector of the generalized covariant derivative with respect to the x's. Then the vector $\overset{0}{D}_j B_i^\alpha$ in SEX$_n$ is normal to X_{n-1} and may be given by [(I.3.36), (I.3.37)]

$$\overset{0}{D}_j B_i^\alpha = -\Omega_{ij} N^\alpha \tag{1.5}$$

where

$$\Omega_{ij} = -(\overset{0}{D}_j B_i^\alpha) N_\alpha \tag{1.6}$$

Furthermore, the tensor Ω_{ij} is the induced tensor on X_{n-1} of the tensor $D_\beta N_\alpha$ in SEX$_n$. That is [(I.3.38)],

$$\Omega_{ij} = (D_\beta N_\alpha) B_i^\alpha B_j^\beta \tag{1.7}$$

On the X_{n-1} of a SEX$_n$ the SE identity (I-4.3) can be written as

$$k_{\alpha\beta}(\Omega_{ik} B_j^\alpha - \Omega_{kj} B_i^\alpha) N^\beta = 0 \tag{1.8}$$

2. THE GENERALIZED FUNDAMENTAL EQUATIONS FOR THE HYPERSUBMANIFOLD OF SEX$_n$

This section is devoted to the derivation of the generalized fundamental equations for the hypermanifold X_{n-1} of SEX$_n$. Here we derive the generalized Gauss formulas, Weingarten equations, and Gauss–Codazzi equations for X_{n-1}.

Theorem 2.1. On X_{n-1} of an SEX_n the generalized coefficients of the second fundamental form Ω_{ij} may be given by

$$\Omega_{ij} = \overset{0}{\Omega}_{ij} - 2k_{(\alpha}{}^{\gamma}X_{\beta)}B_i^{\alpha}B_j^{\beta}N_{\gamma} \tag{2.1}$$

where $\overset{0}{\Omega}_{ij}$ are the coefficients of the second fundamental form with respect to the Christoffel symbols $\{{}_{\lambda}{}^{\nu}{}_{\mu}\}$.

Proof. Substituting (I.2.12) into (1.7) and making use of (1.1) and (1.7), our assertion follows in the following way:

$$\begin{aligned}\Omega_{ij} &= [\partial_{\beta}N_{\alpha} - N_{\gamma}(\{{}_{\alpha}{}^{\gamma}{}_{\beta}\} + 2k_{(\alpha}{}^{\gamma}X_{\beta)} + 2\delta_{[\alpha}{}^{\gamma}X_{\beta]})]B_i^{\alpha}B_j^{\beta} \\ &= \overset{0}{\Omega}_{ij} - 2k_{(\alpha}{}^{\gamma}X_{\beta)}B_i^{\alpha}B_j^{\beta}N_{\gamma}\end{aligned}$$ ■

Remark 2.2. In virtue of (2.1), we note that *the tensor Ω_{ij} is symmetric on X_{n-1} of SEX_n*, while the generalized coefficients $\overset{x}{\Omega}_{ij}$ is not symmetric on a general submanifold X_m of X_n [(I.3.38)].

Theorem 2.3. (The generalized Gauss formulas for SEX_n.) On the X_{n-1} of a SEX_n the following relation holds

$$D_jB_i^{\alpha} = -\overset{0}{\Omega}_{ij}N^{\alpha} + 2k_{(\beta}{}^{\varepsilon}X_{\gamma)}B_i^{\beta}B_j^{\gamma}N_{\varepsilon}N^{\alpha} \tag{2.2}$$

Proof. Substituting (2.1) into (1.5), we have (2.2). ■

In order to prove the generalized Weingarten equations, we need the induced tensors M_j^i of $D_{\beta}N^{\alpha}$ and M_j of $(D_{\beta}N^{\alpha})N_{\alpha}$, respectively, on X_{n-1} of SEX_n:

$$M_j^i = (D_{\beta}N^{\alpha})B_{\alpha}^iB_j^{\beta} \tag{2.3a}$$

$$M_j = (D_{\beta}N^{\alpha})N_{\alpha}B_j^{\beta} = -(D_{\beta}N_{\alpha})N^{\alpha}B_j^{\beta} \tag{2.3b}$$

In the following three theorems, we derive useful representations of the induced tensors M_j^i and M_j.

Theorem 2.3. In an SEX_n the system of equations (I.2.8b) may be given by

$$D_{\omega}g_{\lambda\mu} = 4g_{\lambda[\omega}X_{\mu]} \tag{2.4}$$

which can be split into

$$D_{\omega}h_{\lambda\mu} = 2(X_{(\lambda}g_{\mu)\omega} - h_{\lambda\mu}X_{\omega}) \tag{2.5a}$$

$$D_{\omega}k_{\lambda\mu} = 2(X_{[\mu}g_{\lambda]\omega} - k_{\lambda\mu}X_{\omega}) \tag{2.5b}$$

Furthermore, in an SEX_n we also have

$$D_{\omega}h^{\lambda\nu} = -2h^{\lambda\alpha}h^{\nu\beta}(X_{(\alpha}g_{\beta)\omega} - h_{\alpha\beta}X_{\omega}) \tag{2.6}$$

Proof. Substitution of (I.2.9) into (I.2.8b) gives (2.4). Equations (2.5a) and (2.5b) follows from (2.4) and

$$D_\omega h_{\lambda\mu} = D_\omega g_{(\lambda\mu)}, \qquad D_\omega k_{\lambda\mu} = D_\omega g_{[\lambda\mu]}$$

On the other hand, if we differentiate both sides of (I.2.4) with respect to y^ν and substitute (2.5a), we get

$$h_{\lambda\mu} D_\omega h^{\lambda\nu} = -h^{\lambda\nu} D_\omega h_{\lambda\mu} = -2h^{\lambda\nu}(X_{(\lambda} g_{\mu)\omega} - h_{\lambda\mu} X_\omega)$$

The relation (2.6) follows immediately by multiplying by $h^{\alpha\mu}$ on both sides of the above equation. ■

Theorem 2.4. The induced tensor M_j^i is given by

$$M_j^i = -2h^{im} X_{(\alpha} k_{\beta)\gamma} N^\alpha B_m^\beta B_j^\gamma - \delta_j^i X_\alpha N^\alpha + h^{im}\Omega_{mj} \tag{2.7}$$

Proof. Equation (2.3a) gives

$$\begin{aligned} M_j^i &= (D_\beta(h^{\alpha\gamma} N_\gamma)) B_\alpha^i B_j^\beta \\ &= (D_\beta h^{\alpha\gamma}) N_\gamma B_\alpha^i B_j^\beta + h^{\alpha\gamma}(D_\beta N_\gamma) B_\alpha^i B_j^\beta \end{aligned} \tag{2.8}$$

Substituting (2.6) into (2.8) and making use of (1.3), (I-3.18), (1.1), and (1.7), we have (2.7). ■

Theorem 2.5. The induced vector M_j is given by

$$M_j = X_\alpha B_j^\alpha - X_{(\alpha} k_{\beta)\gamma} N^\alpha N^\beta B_j^\gamma \tag{2.9}$$

Proof. Generalized covariant differentiation of both sides of the last relation of (1.1) with respect to x^j gives

$$(D_\gamma h_{\alpha\beta}) N^\alpha N^\beta B_j^\gamma + 2h_{\alpha\beta}(D_\gamma N^\alpha) N^\beta B_j^\gamma = 0 \tag{2.10}$$

Our representation (2.9) immediately follows by substituting (2.5a) and (2.3b) into (2.10) and making use of (1.1). ■

Now, we are ready to prove the following generalized Weingarten equations.

Theorem 2.6a. (The first representation of the generalized Weingarten equations in SEX_n.) On the X_{n-1} of an SEX_n the following relation holds:

$$\begin{aligned} \overset{0}{D}_j N^\alpha &= X_{(\varepsilon} k_{\beta)\gamma}(N^\alpha N^\beta - 2h^{\alpha\beta}) N^\varepsilon B_j^\gamma + h^{im}\Omega_{mj} B_i^\alpha \\ &\quad - X_\beta N^\beta B_j^\alpha + X_\beta N^\alpha B_j^\beta \end{aligned} \tag{2.11}$$

Proof. Substituting (1.2a) for δ_γ^α into

$$\overset{0}{D}_j N^\alpha = (D_\beta N^\alpha) B_j^\beta = (\delta_\gamma^\alpha D_\beta N^\gamma) N_j^\beta$$

and making use of (2.3a), (2.3b), and (I.3.15), we have

$$\overset{0}{D}_j N^\alpha = M_j^i B_i^\alpha + M_j N^\alpha \tag{2.12}$$

Our assertion (2.11) immediately follows by substituting (2.7) and (2.9) into (2.12) and making use of (1.4b). ■

Theorem 2.6b. (The second representation of the generalized Weingarten equations in SEX_n.) On the X_{n-1} of an SEX_n the following relation holds:

$$\overset{0}{D}_j N_\alpha = \Omega_{ij} B^i_\alpha + X_{(\beta} k_{\varepsilon)\gamma} N_\alpha N^\beta N^\varepsilon B^\gamma_j - X_\beta N_\alpha B^\beta_j \tag{2.13}$$

Proof. Substituting (2.5a) and (2.11) into

$$\overset{0}{D}_j N_\alpha = \overset{0}{D}_j (h_{\alpha\beta} N^\beta) = h_{\alpha\beta} \overset{0}{D}_j N^\beta + (D_\gamma h_{\alpha\beta}) N^\beta B^\gamma_j$$

and making use of (1.1), we have (2.13). ■

In order to derive the generalized Gauss-Codazzi equations, we need the following curvature tensors of SEX_n and its hypersubmanifold X_{n-1}:

$$R_{\omega\mu\lambda}{}^\nu = 2(\partial_{[\mu} \Gamma^\nu_{|\lambda|\omega]} + \Gamma^\nu_{\alpha[\mu} \Gamma^\alpha_{|\lambda|\omega]}) \tag{2.14}$$

$$R_{ijk}{}^m = 2(\partial_{[j} \Gamma^m_{|k|i]} + \Gamma^m_{p[j} \Gamma^p_{|k|i]}) \tag{2.15}$$

Theorem 2.7. (The generalized Gauss-Codazzi equations in SEX_n.) On the X_{n-1} of an SEX_n the curvature tensors defined by (2.14) and (2.15) are involved in the following identities:

$$\begin{aligned} R_{ijk}{}^p = {} & R_{\beta\gamma\varepsilon}{}^\alpha B^p_\alpha B^\varepsilon_k B^\gamma_j B^\beta_i + 2(\Omega_{m[j} \Omega_{|k|i]} h^{mp} B^\alpha_p \\ & + X_\beta N^\beta \Omega_{k[j} \delta^p_{i]} + k^\alpha_\gamma X_\beta N^\beta \Omega_{k[i} B^\gamma_{j]} B^p_\alpha) \end{aligned} \tag{2.16}$$

$$2\overset{0}{D}_{[k} \Omega_{|i|j]} = R_{\beta\gamma\varepsilon}{}^\alpha N_\alpha B^\beta_k B^\gamma_j B^\varepsilon_i + 2(X_\beta \Omega_{i[k} B^\beta_{j]} + 2\Omega_{i[k} X_{j]}) \tag{2.17}$$

Proof. In virtue of (1.5), (2.14), (2.15), and

$$\overset{0}{D}_j B^\alpha_i = B^\alpha_{ij} + \Gamma^\alpha_{\beta\gamma} B^\beta_i B^\gamma_j - \Gamma^k_{ij} B^\alpha_k$$

we have

$$\begin{aligned} 2\overset{0}{D}_{[k} \overset{0}{D}_{j]} B^\alpha_i &= 2[\partial_{[k} (\overset{0}{D}_{j]} B^\alpha_i) - \Gamma^m_{[jk]} (\overset{0}{D}_m B^\alpha_i) - \Gamma^m_{i[k} (\overset{0}{D}_{j]} B^\alpha_m) + \Gamma^\alpha_{\beta\gamma} (\overset{0}{D}_{[j} B^\beta_{|i|}) B^\gamma_{k]}] \\ &= -R_{\varepsilon\gamma\beta}{}^\alpha B^\beta_i B^\gamma_j B^\varepsilon_k + R_{kji}{}^m B^\alpha_m + 4\Omega_{i[j} X_{k]} N^\alpha \end{aligned} \tag{2.18}$$

where use of the relation

$$S_{jk}{}^m = 2\delta_{[j}{}^m X_{k]}$$

has been made in the above lengthy calculation. On the other hand, the relations (1.5) and (2.11) give

$$\overset{0}{D}_{[k}\overset{0}{D}_{j]}B_i^\alpha = -2(\overset{0}{D}_{[k}\Omega_{|i|j]})N^\alpha - 2\Omega_{i[j}\overset{0}{D}_{k]}N^\alpha$$

$$= -2(\overset{0}{D}_{[k}\Omega_{|i|j]} + X_\beta\Omega_{i[j}B_{k]}^\beta)N^\alpha$$

$$-2X_{(\varepsilon}k_{\beta)\gamma}N^\varepsilon N^\beta B_{[k}^\gamma\Omega_{|i|j]}N^\alpha + 2\Omega_{m[j}\Omega_{|i|k]}h^{mp}B_p^\alpha$$

$$+2X_\beta N^\beta\Omega_{i[j}B_{k]}^\alpha + 4h^{\alpha\beta}X_{(\varepsilon}k_{\beta)\gamma}N^\varepsilon\Omega_{i[j}B_{k]}^\gamma \tag{2.19a}$$

In virtue of the SE identity (1.8) and the symmetry of Ω_{ij}, the second and the fifth terms of the last equation of (2.19a) are

$$\text{Second Term} = 0 \tag{2.19b}$$

$$\text{Fifth Term} = -2X_\varepsilon k_\gamma{}^\alpha N^\varepsilon\Omega_{i[j}B_{k]}^\gamma + 2X^\alpha k_{\varepsilon\gamma}\Omega_{i[j}B_{k]}^\gamma N^\varepsilon$$

$$= 2k_\gamma{}^\alpha X_\beta N^\beta\Omega_{i[k}B_{j]}^\gamma \tag{2.19c}$$

Comparing (2.18) and (2.19), one finally gets

$$R_{kji}{}^m B_m^\alpha = R_{\varepsilon\gamma\beta}{}^\alpha B_i^\beta B_j^\gamma B_k^\varepsilon + 2(-\overset{0}{D}_{[k}\Omega_{|i|j]} + X_\beta\Omega_{i[k}B_{j]}^\beta + 2\Omega_{i[k}X_{j]})N^\alpha$$

$$+2(\Omega_{m[j}\Omega_{|i|k]}h^{mh}B_h^\alpha + X_\beta N^\beta\Omega_{i[j}B_{k]}^\alpha + k_\gamma{}^\alpha X_\beta N^\beta\Omega_{i[k}B_{j]}^\gamma) \tag{2.20}$$

Making use of (I.3.16), the identity (2.16) follows by multiplying by B_α^p on both sides of (2.20) and interchanging the indices i and k. On the other hand, multiplying by N_α on both sides of (2.20) and using the SE identity (1.8), we have (2.17). ■

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